Yang-Lee circle theorem for an ideal pseudospin-1/2 Bose gas in an external magnetic field

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The Yang-Lee circle theorem is extended to an ideal pseudospin-1/2 Bose gas in an external magnetic field. It is found that the zeros of the canonical partition function are located on the unit circle in the complex activity plane if the temperature is above the critical temperature of ideal Bose-Einstein condensation. No zeros exist if the temperature is below the critical temperature.

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I. INTRODUCTION

Recent experiments [1,2] on dilute quantum gases of alkali-metal atoms have produced spin-1 and pseudospin-1/2 Bose gases, respectively. One may raise an interesting question: Does the celebrated Yang-Lee circle theorem hold for Bose gases? As is well known, the circle theorem was proved originally for the ferromagnetic Ising model [3,4]. The theorem was further extended to many ferromagnetic lattice systems, such as the quantum Heisenberg model [5], the classical XY and Heisenberg models [6], the high-spin Ising model [7], the multiple-spin-interaction Ising model [5,8], and some continuous spin systems [9]. The circle theorem was further extended to noncircular regions [10]. To our knowledge, the circle theorem has never been extended to a continuous gas. In this paper the circle theorem is extended to an ideal Bose gas.

This paper is organized as follows. In Sec. II, an ideal pseudospin-1/2 Bose is introduced and solved in the grand canonical ensemble. The canonical partition function is derived. In Sec. III, the numerical results of Yang-Lee zeros are presented. In Sec. IV, four lemmas and the circle theorem are proved. In Sec. V, the zero density is determined. In Sec. VI, the absence of Yang-Lee zeros in the Bose-Einstein region is proved. In Sec. VII, a discussion and summary are given.

II. MODEL

Let us consider an ideal Bose gas composed of pseudospin-1/2 Bose atoms. The atoms possess LS coupling, with a total angular momentum $J=1/2$. Therefore, the energy spectrum of an atom in the presence of an external magnetic field $\vec{H}$ is given by

$$\epsilon(\vec{p},M_J)=\frac{p^2}{2m}-\mu_B g M_J H, \quad M_J=-1/2,1/2, \tag{1}$$

where μ_B is the Bohr magneton and g the Landé factor. The unit of H is chosen as $\mu_B g/2=1$. This model is exactly solvable in the grand canonical ensemble. As shown by Landau and Lifshitz [11], the thermodynamic potential is given by

$$\Omega(\mu)=\Omega_0(\mu+H)+\Omega_0(\mu-H), \tag{2}$$

where $\Omega_0(\mu)$ is the thermodynamic potential of a spinless ideal Bose gas in the absence of a magnetic field. μ is the chemical potential, and $\mu\leqslant -H$ is required. μ is determined by

$$N=C(d)\int_0^\infty\left[\frac{1}{e^{(p^2/2m-\mu+H)/k_BT}-1}+\frac{1}{e^{(p^2/2m-\mu-H)/k_BT}-1}\right]p^{d-1}dp, \tag{3}$$

where $C(d)=2V\pi^{d/2}h^{-d}/\Gamma(d)$, and h is the Planck constant.

The net magnetic moment acquired by the gas is

$$M=-(\partial\Omega/\partial H)_{T,V,\mu} = C(d)\int_0^\infty\left[\frac{1}{e^{(p^2/2m-\mu-H)/k_BT}-1}-\frac{1}{e^{(p^2/2m-\mu+H)/k_BT}-1}\right]p^{d-1}dp. \tag{4}$$

In $d\geqslant 3$, the gas undergoes a Bose-Einstein condensation in the presence of an external magnetic field H at a critical temperature $T_c(H)$ as the chemical potential reaches the maximum value $\mu_{max}=-H$. $T_c(H)$ is given by

$$N=C(d)\int_0^\infty\left[\frac{1}{e^{p^2/2mk_BT_c}-1}+\frac{1}{e^{(p^2/2m+2H)/k_BT_c}-1}\right]p^{d-1}dp. \tag{5}$$

For $T>T_c(H)$, as $H\to 0$, we obtain

$$M/H=C(d)\beta\int_0^\infty\frac{e^{\beta(p^2/2m-\mu)}}{[e^{\beta(p^2/2m-\mu)}-1]^2}p^{d-1}dp, \tag{6}$$

where $\beta=1/k_BT$. Therefore the gas is paramagnetic.

For $T<T_c(H)$, the number of bosons condensed in the ground state is

$$N_0(H)=N\left[1-(T/T_c(H))^{d/2}\times\frac{\zeta(d/2)+\sum_{l=1}^{\infty} lb_l(e^{-2H/k_BT})^l}{\zeta(d/2)+\sum_{l=1}^{\infty} lb_l(e^{-2H/k_BT_c(H)})^l}\right], \qquad (7)$$

where $b_l=l^{-1-d/2}$ [12]. The magnetic moment is

$$M(H)=N-2C(d)\int_0^{\infty}\frac{1}{e^{\beta(p^2/2m+2H)}-1}p^{d-1}dp. \qquad (8)$$

As $H\to 0$, $M(H)$ exhibits a scaling law

$$M(H)\sim H^{d/2-1}. \qquad (9)$$

The spontaneous magnetization is $M(H=0)=N_0(H=0)=N[1-(T/T_c(0))^{d/2}]$. The derivatives of M with respect to H are $(\partial^n M/\partial H^n)_T(H=0)$ finite for $n<d/2-1$ and $(\partial^n M/\partial H^n)_T(H=0)=\infty$ for $n\geqslant d/2-1$. Thus the gas undergoes a magnetic phase transition at $H=0$ and $T<T_c(H=0)$. For $d=1$ and 2, no Bose-Einstein condensation, and the magnetic phase transition exists at a finite temperature.

From Eq. (2), we obtain the grand partition function of the gas,

$$\Xi(\mu)=\Xi_0(\mu+H)\Xi_0(\mu-H), \qquad (10)$$

where $\Xi_0(\mu)$ is the grand partition function of a spinless ideal Bose gas. For $T>T_c(H)$, using $\Xi(\mu)=\sum_{n=0}^{\infty}\exp(\beta\mu)Q(N)$, expanding Eq. (10) as a power series in $z=\exp(\beta\mu)$, and equating the respective coefficients, we obtain the canonical partition function of the gas,

$$Q(N)=e^{N\beta H}\sum_{n=0}^{N} x^n Q_0(n)Q_0(N-n), \qquad (11)$$

where $x=\exp(-2\beta H)$, and $Q_0(n)$ is the canonical partition function of an ideal Bose gas of n spinless bosons, determined by [12]

$$\sum_{n=0}^{\infty} z^n Q_0(n)=\exp\left[\frac{V}{\lambda^d}\sum_{l=1}^{\infty} z^l b_l\right], \qquad (12)$$

where $\lambda=h/(2\pi mk_BT)^{1/2}$ is the thermal wavelength. Expanding Eq. (12) as a power series in z, and equating the respective coefficients, we obtain [13]

$$Z_0(0)=1,\quad Z_0(1)=V,\quad Z_0(2)=Z_0(1)^2+2\lambda^d Vb_2,$$
$$Z_0(3)=3Z_0(1)Z_0(2)-2Z_0(1)^3+3!\lambda^{2d}Vb_3,\ \dots, \qquad (13)$$

where $Z_0(l)=l!\lambda^{ld}Q_0(l)$.

III. NUMERICAL RESULTS

We have numerically checked the zeros of $Q(N)$ for various values of N, d, and V/λ^d. Here we list only a few.

A. $N=5$

(1) $V/\lambda^d=1000$ $(d=3)$. The zeros are

$$x=-1,-0.999351\pm 0.0360326i,-0.997117\pm 0.0758817i. \qquad (14)$$

(2) $V/\lambda^d=10$ $(d=3)$. The zeros are

$$x=-1,-0.938546\pm 0.345153i,-0.736462\pm 0.676479i. \qquad (15)$$

(3) $V/\lambda^d=0.1$ $(d=3)$. The zeros are

$$x=-1,-0.356202\pm 0.934409i,0.767583\pm 0.640949i. \qquad (16)$$

B. $N=7$

(1) $V/\lambda^d=10$ $(d=3)$. The zeros are

$$x=-1,-0.955818\pm 0.293959i,$$
$$-0.818582\pm 0.57439i,-0.56515\pm 0.824988i. \qquad (17)$$

(2) $V/\lambda^d=1$ $(d=3)$. The zeros are

$$x=-1,-0.764154\pm 0.645034i,$$
$$-0.135269\pm 0.990809i,0.639424\pm 0.768854i. \qquad (18)$$

(3) $V/\lambda^d=0.1$ $(d=3)$. The zeros are

$$x=-1,-0.642172\pm 0.76656i,$$
$$0.183238\pm 0.983069i,0.88468\pm 0.466198i. \qquad (19)$$

C. $N=20$

(1) $V/\lambda^d=1$ $(d=3)$. The zeros are

$$x=-0.989538\pm 0.144274i,-0.906964\pm 0.421207i,$$
$$-0.747762\pm 0.663967i,-0.523573\pm 0.851981i,$$
$$-0.251248\pm 0.967923i,0.0478592\pm 0.998854i,$$
$$0.348772\pm 0.937207i,0.62388\pm 0.781521i,$$
$$0.843785\pm 0.536681i,0.977177\pm 0.212425i. \qquad (20)$$

(2) $V/\lambda^d=0.2$ $(d=3)$. The zeros are

$$x=-0.988093\pm0.153857i,-0.894482\pm0.447104i,$$
$$-0.715871\pm0.698233i,-0.468752\pm0.88333i,$$
$$-0.176105\pm0.984371i,0.134572\pm0.990904i,$$
$$0.433626\pm0.901093i,0.691845\pm0.722046i,$$
$$0.883021\pm0.469333i,0.986201\pm0.165552i. \quad (21)$$

From these examples, we see clearly that the zeros are located on the unit circle $|x|=1$.

IV. CIRCLE THEOREM

A. Four lemmas

Before we prove the circle theorem, let us state four lemmas.

Lemma 1: If $Re^{i\phi}$ is a root of $Q(N)$, then $(1/R)e^{i\phi}$ is another root (R and ϕ are real).

This result is evident because of the symmetry property $Q(N,x)=Q(N,x^{-1})$.

Lemma 2: For an odd N, $x=-1$ is a root of $Q(N)$.

This result is evident.

Lemma 3: For $N=1$, 2, and 3, the zeros of $Q(N)$ are located on the unit circle $|x|=1$.

Proof. For $N=1$, we have

$$Q(1)=e^{\beta H}Q_0(0)Q(1)(1+x). \quad (22)$$

Hence the root of $Q(1)$ is $x=-1$.

For $N=2$, we have

$$Q(2)=e^{2\beta H}[(1+x^2)Q_0(0)Q_0(2)+xQ_0(1)^2]. \quad (23)$$

The roots of $Q(2)$ are

$$x=-\frac{Q_0(1)^2}{2Q_0(2)}\pm\sqrt{\left[\frac{Q_0(1)^2}{2Q_0(2)}\right]^2-1}. \quad (24)$$

From Eq. (13), we obtain

$$Q_0(2)=\frac{Q_0(1)^2}{2}+\frac{V}{\lambda^d 2^{1+d/2}}. \quad (25)$$

It follows that $|x|=1$.

For $N=3$, we have

$$Q(3)=e^{3\beta H}[(1+x^3)Q_0(0)Q(3)+(x+x^2)Q_0(1)Q_0(2)]. \quad (26)$$

The roots of $Q(3)$ are

$$x=-1,-\frac{Q_0(1)Q_0(2)-Q_0(3)}{2Q_0(3)}\pm\sqrt{\left[\frac{Q_0(1)Q_0(2)-Q_0(3)}{2Q_0(3)}\right]^2-1}. \quad (27)$$

From Eq. (13) we obtain

$$\frac{Q_0(1)Q_0(2)-Q_0(3)}{2Q_0(3)}=\frac{1-3b_3(\lambda^d/V)^2}{1+6b_2(\lambda^d/V)+6b_3(\lambda^d/V)^2}, \quad (28)$$

with the absolute magnitude less than 1. Hence we have $|x|=1$.

Lemma 4: In the classical limit, the zeros of $Q(N)$ are located on the unit circle $x=-1$.

Proof. In the classical limit, Eq. (11) becomes

$$Q(N)=\frac{e^{N\beta H}}{N!}(V/\lambda^d)^N(1+x)^N. \quad (29)$$

Therefore, the zeros are $x=-1$.

B. Circle theorem

For $N=4$, an analytic formula of the roots of $Q_0(4)$ is available. However, its mathematical expression is so complicated that it is impossible to prove $|x|=1$ analytically. For $N\geqslant5$, no analytic root formula is available. Therefore for $N\geqslant4$, we have to seek a new method to prove the circle theorem.

Long ago it was shown by Uhlenbeck and Gropper [13,14,17] that the canonical partition function of two spinless noninteracting bosons is equal to that of two interacting classical particles with the attractive potential $u_s(r)$ at the same m, T and V. $u_s(r)$ is given by

$$u_s(r)=-k_BT\ln[1+\exp(-2\pi r^2/\lambda^2)]. \quad (30)$$

We try to extend this result to the N-particle case: Let us introduce a fictitious classical interacting system of N particles, with the two-body potential $u(r)$. The classical cluster integrals of this system are $b_l=l^{-1-d/2}$ ($l=2,\ldots,N$). This means that the canonical partition function of n ($2\leqslant n\leqslant N$) noninteracting spinless bosons is equal to that of n interacting classical particles with the two-body potential $u(r)$, at the same m, V, and T. Since b_l is independent of temperature, $u(r)$ is a function of r/λ.

The condition that $u(r)$ be real is too strong and may not be always satisfied. Since $u(r)$ is fictitious, we may relax this condition and require that $\exp(-\beta u)$ be real. This means that the negative Boltzmann weights may exist. On the other hand, the symmetry property of the wave functions of a Bose system leads to a statistically effective attraction between the bosons [11]. Hence $u(r)$ must be negative for the great majority of $0\leqslant r<\infty$.

In order to confirm this, let us consider the case $N=3$, and use the fitting function

$$e^{-\beta u(r)}=1+\alpha_1 e^{-2\pi r^2/\lambda^2}+\alpha_2(e^{-2\pi r^2/\lambda^2})^2, \quad (31)$$

where α_1 and α_2 are constants, determined by $b_l=l^{-1-d/2}$ ($l=2$ and 3). The results are $\alpha_1=2.01831$ and $\alpha_2=-2.88022$ for $d=3$, and $\alpha_1=4.12677$ and $\alpha_2=-6.25354$ for $d=2$. We see that $u(r)$ is positive or complex for smaller r and $u(r)$ is negative for larger r.

This fictitious potential is not uniquely determined. There exist an infinity of possible fitting functions of $u(r)$ that satisfy the requirements stated. For our latter purpose we may choose $u(r)$ in a such way that $\exp(-\beta u)\leqslant -1$ for smaller r and $\exp(-\beta u)\geqslant 1$ for larger r.

Using the above results, Eq. (11) becomes

$$Q(N)=\frac{e^{N\beta H}}{N!\lambda^{dN}}\sum_{l=0}^{N}\frac{N!}{l!(N-l)!}Z_0(l)Z_0(N-l)x^l, \quad (32)$$

with

$$Z_0(l)=\int d^dr_1\cdots d^dr_l\prod_{1\leqslant i<j\leqslant l}e^{-\beta u_{ij}}, \quad (33)$$

where $u_{ij}=u(r_{ij})$.

The main results of this paper are as follows.

Theorem 1: Equations (32) and (33) imply that

$$Q(N)=\frac{e^{N\beta H}}{N!\lambda^{dN}}\int d^dr_1\cdots d^dr_N \times\left[\prod_{1\leqslant i<j\leqslant N}e^{-\beta u_{ij}}\right]Y_N(x,\ldots,x), \quad (34)$$

where $Y_N(x_1,\ldots,x_N)$ is the Yang-Lee polynomial [15,16],

$$Y_N(x_1,\ldots,x_N)=\sum_S x_{i_1}\cdots x_{i_s}\left(\prod_{i\in S}\prod_{j\in S'}A_{ij}\right), \quad (35)$$

where $A_{ij}=e^{\beta u_{ij}}$, satisfying $A_{ij}=A_{ji}$, $-1\leqslant A_{ij}\leqslant 1$. The summation is over all subsets $S=\{i_1,\ldots,i_s\}$ of the set $\Delta_N=\{1,\ldots,N\}$, and $S'=\{j_1,\ldots,j_{N-s}\}$ is the complement of S in Δ_N.

Proof. Define a polynomial in variables $x_1,\ldots,x_N$:

$$\mathcal{B}_N(x_1,\ldots,x_N)=\sum_{S\in\Delta_N}(x_{i_1}\cdots x_{i_s})Z_0(s)Z_0(N-s). \quad (36)$$

Thus Eq. (32) becomes

$$Q(N)=\frac{e^{N\beta H}}{N!\lambda^{dN}}\mathcal{B}_N(x,\ldots,x). \quad (37)$$

The polynomial $\mathcal{B}_N(x_1,\ldots,x_N)$ may be expressed as

$$\begin{aligned}\mathcal{B}_N(x_1,\ldots,x_N)&=\sum_{S\in\Delta_N}(x_{i_1}\cdots x_{i_s})\int d^dr_{i_1}\cdots d^dr_{i_s}\int d^dr_{j_1}\cdots d^dr_{j_{N-s}}\left[\prod_{\{n<m\}\in S}e^{-\beta u_{nm}}\right]\left[\prod_{\{n'<m'\}\in S'}e^{-\beta u_{n'm'}}\right]\\&=\int d^dr_1\cdots d^dr_N\left[\prod_{1\leqslant i<j\leqslant N}e^{-\beta u_{ij}}\right]\sum_{S\in\Delta_N}(x_{i_1}\cdots x_{i_s})\left[\prod_{i\in S}\prod_{j\in S'}e^{\beta u_{ij}}\right]\\&=\int d^dr_1\cdots d^dr_N\left[\prod_{1\leqslant i<j\leqslant N}e^{-\beta u_{ij}}\right]Y_N(x_1,\ldots,x_N). \quad (38)\end{aligned}$$

This completes the proof.

Theorem 2: Equation (34) implies that the zeros of $Q(N)$ are located on the unit circle $|x|=1$.

Proof. According to the Yang-Lee lemma [3,15], Eq. (34) becomes

$$\int d^dr_1\cdots d^dr_N\left[\prod_{1\leqslant i<j\leqslant N}e^{-\beta u_{ij}}\right]\times\prod_{l=1}^{N}(x-e^{i\phi_l})=Z_0(N)\prod_{l=1}^{N}(x-\eta_l), \quad (39)$$

where $e^{i\phi_l}$ are the roots of the Yang-Lee polynomial $Y_N(x,\ldots,x)$, and η_l are the roots of $Q(N)$. From Eq. (39), we obtain

$$\sum_{l=1}^{N}\int d^dr_1\cdots d^dr_N\left[\prod_{1\leqslant i<j\leqslant N}e^{-\beta u_{ij}}\right]e^{i\phi_l}=Z_0(N)\sum_{l=1}^{N}\eta_l. \quad (40)$$

We see that η_l are determined by

$$\eta_l=\frac{1}{Z_0(N)}\int d^dr_1\cdots d^dr_N\left[\prod_{1\leqslant i<j\leqslant N}e^{-\beta u_{ij}}\right]e^{i\phi_l}\equiv|\eta_l|e^{i\theta_l}. \quad (41)$$

According to Lemma 1, $|\eta_l|^{-1}e^{i\theta_l}$ is another root of $Q(N)$. Since there exists one to one correspondence between the roots of $Q(N)$ and the roots of the Yang-Lee polynomial $Y_N(x,\ldots,x)$, this is impossible. Hence we have $|\eta_l|=|\eta_l|^{-1}$, which implies $|\eta_l|=1$. This completes the proof.

Let us make a comment on our method. By introducing a two-body potential, we transform the ideal spinless Bose gas into a representation of the classical interacting gas. In this way, $Q(N)$ can be expressed as a spatial integral of the Yang-Lee polynomial, and the circle theorem follows. Since the zeros of $Q(N)$ depend on only $Q_0(n)$, the zeros are independent of the two-body potential $u(r)$.

V. ZERO DENSITY

Recently, the zero density of the Ising ferromagnet $FeCl_2$ was obtained experimentally by analyzing its isothermal magnetization data [18]. Here we determine the zero density of our model.

The canonical partition function of the gas is given by

$$Q(N)=e^{N\beta H}Q_0(N)\prod_{l=1}^{N}(x-e^{i\theta_l}). \tag{42}$$

In the thermodynamic limit, the free energy is given by

$$\begin{aligned}F&=-k_BT\ln Q(N)\\&=-NH-k_BT\ln Q_0(N)-Nk_BT\\&\quad\times\int_0^\pi g(\theta)\ln(x^2-2x\cos\theta+1)d\theta,\end{aligned} \tag{43}$$

where $g(\theta)$ is the zero density, with the normalization condition

$$\int_0^\pi g(\theta)d\theta=1/2. \tag{44}$$

Thus the normalized magnetization may be expressed as

$$\begin{aligned}I(x)&=M(H)/N\\&=-\frac{1}{N}\frac{\partial F}{\partial H}=2(1-x^2)\int_0^\pi\frac{g(\theta)}{x^2-2x\cos\theta+1}d\theta.\end{aligned} \tag{45}$$

From Eqs. (3) and (4), we obtain

$$\frac{N\lambda^d}{V}=\sum_{l=1}^{\infty}lb_lz^l\cosh(\beta Hl) \tag{46}$$

and

$$\frac{M\lambda^d}{V}=\sum_{l=1}^{\infty}lb_lz^l\sinh(\beta Hl). \tag{47}$$

The virial expansion of M is

$$\frac{M\lambda^d}{V}=\sum_{l=1}^{\infty}a_l(x)\left(\frac{N\lambda^d}{V}\right)^l, \tag{48}$$

with $a_l(x^{-1})=-a_l(x)$. Subsituting Eq. (46) into Eq. (48), and comparing with Eq. (47), we obtain

$$a_1=\frac{(x-1)}{(1+x)},\quad a_2=8b_2\frac{x(x-1)}{(1+x)^3},\ \ldots. \tag{49}$$

In the high field limit, $I(x)$ may be expanded as a power series of x,

$$I(x)=1+2\pi\sum_{n=1}^{\infty}g_nx^n, \tag{50}$$

with

$$g_n=(2/\pi)\int_0^\pi g(\theta)\cos(n\theta)d\theta. \tag{51}$$

Then the zero density is given by [19]

$$\begin{aligned}g(\theta)&=(1/2\pi)\lim_{r\to1^-}\operatorname{Re}I(re^{i\theta})\\&=(1/2\pi)\frac{V}{N\lambda^d}\sum_{l=1}^{\infty}\left(\frac{N\lambda^d}{V}\right)^l\lim_{r\to1^-}\operatorname{Re}a_l(re^{i\theta}).\end{aligned} \tag{52}$$

VI. BOSE-EINSTEIN CONDENSATION REGION

For $T<T_c(H)$, the chemical potential is $\mu=-H$. So the free energy is

$$F=\Omega+\mu N=\Omega_0(0)+\Omega_0(-2H)-HN. \tag{53}$$

Therefore, using $F=-k_BT\ln Q$, we obtain, in the thermodynamic limit, the canonical partition function

$$\begin{aligned}Q(N)&=e^{N\beta H}\Xi_0(0)\Xi_0(-2H)\\&=e^{N\beta H}\Xi_0(0)\prod_{\vec p}\frac{1}{1-x\exp(-\beta p^2/2m)}.\end{aligned} \tag{54}$$

On the other hand, the grand partition function of an ideal spinless Bose gas is given by

$$\Xi_0(\mu)=\sum_{n=0}^{\infty}Q_0(n)z^n=\prod_{\vec p}\frac{1}{1-z\exp(-\beta p^2/2m)}. \tag{55}$$

Since, for an ideal Bose gas, no hard core exists, the grand partition function $\Xi_0(\mu)$ is not a polynomial of the fugacity. Hence no zeros of $\Xi_0(\mu)$ exist [20].

It is interesting to note from Eqs. (54) and (55) that $Q(N)$ as a function of x is identical to $\Xi_0(\mu)$ as a function of z, besides a factor. Therefore, we deduce that in the Bose-Einstein condensation region, no zeros of $Q(N)$ exist.

VII. DISCUSSION AND CONCLUSION

We have introduced an ideal pseudospin-1/2 Bose gas model composed of atoms that possess LS coupling with the total angular momentum $J=1/2$. The model is exactly solvable in the grand canonical ensemble. For $d\geqslant3$, there exists Bose-Einsein condensation even in an external magnetic field. Below the critical temperature, spontaneous magnetization exists. The gas undergoes a phase transition from a paramagnetic state to a ferromagnetic state. Above the critical temperature, Yang-Lee zeros of the canonical partition function exist. It is found that in this case the Yang-Lee circle theorem holds. The circle theorem asserts that a magnetic phase transition is possible only at $H=0$, which is consistent with the exact solution. Below the critial temperature, in the thermodynamic limit, no Yang-Lee zeros exist. For $d=1$ and 2, the circle theorem holds in the whole range of temperature.

Let us compare our model with ferromagnetic lattice models, such as the Ising model. The Ising model is described by a canonical ensemble. Our model is di-scribed by a grand canonical ensemble and also by a canonical ensemble. The zeros of the canonical partition function of the ferromagnetic Ising model exist, either above or below the critical temperature. The thermodynamic properties are determined by Yang-Lee zeros. In the grand canonical ensemble, no Yang-Lee zeros of the grand partition function of our model exist. In the canonical ensemble, above the critical temperature of the Bose-Einstein condensation, the zeros of the canonical partition function of our model exist. The thermodynamic properties are determined by Yang-Lee zeros. However, below the critical temperature, no zeros of the canonical partition function of our model exist. Furthermore, the magnetic phase transition in the ferromagnetic Ising model is related to the Yang-Lee singularity (the positive real Yang-Lee zeros). Above the critical temperature, the magnetic phase transition in our model is also related to the Yang-Lee singularity [the positive real Yang-Lee zeros of $Q(N)$]. However, below the critical temperature, the magnetic phase transition is related to the Bose-Einstein singularity ($[1-x\exp(-\beta p^2/2m)]^{-1}=\infty$ when $x=1(H=0)$ and $p=0$). The asymmetry of the singularity property is caused by ideal Bose-Einstein condensation.

[1] J. Stenger *et al.*, Nature (London) **396**, 345 (1998).

[2] M.R. Matthews *et al.*, Phys. Rev. Lett. **83**, 2498 (1999).

[3] C.N. Yang and T.D. Lee, Phys. Rev. **87**, 404 (1952); T.D. Lee and C.N. Yang, *ibid.* **87**, 410 (1952).

[4] X.Z. Wang and J.S. Kim, Phys. Rev. E **57**, 5013 (1998); X.Z. Wang, Commun. Theor. Phys. (to be published).

[5] M. Suzuki and M.E. Fisher, J. Math. Phys. **12**, 235 (1971).

[6] F. Dunlop and C.M. Newman, Commun. Math. Phys. **44**, 223 (1975).

[7] R.B. Griffiths, J. Math. Phys. **10**, 1559 (1969).

[8] X.Z. Wang and J.S. Kim, Phys. Rev. E **58**, 4174 (1998).

[9] B. Simon and R.B. Griffiths, Commun. Math. Phys. **33**, 145 (1973).

[10] D. Ruelle, Phys. Rev. Lett. **26**, 303 (1971).

[11] L.D. Landau and E.M. Lifshitz, *Statistical Physics* (Pergamon, New York, 1980), Part 1, Secs. 59 and 116.

[12] R.M. Ziff, G.E. Uhlenbeck, and M. Kac, Phys. Rep. **32**, 169 (1977).

[13] R.K. Pathria, *Statistical Physics* (Pergamon, New York, 1972), Secs. 5.5 and 9.4.

[14] K. Huang, *Statistical Physics* (Wiley, New York, 1963), Sec. 10.2.

[15] D. Ruelle, *Statistical Mechanics: Rigorous Results* (Benjamin, New York, 1969), Sec.5.1.

[16] B. Beauzamy, Commun. Math. Phys. **182**, 177 (1996).

[17] G.E. Uhlenbeck and L. Gropper, Phys. Rev. **41**, 79 (1932).

[18] Ch. Binek, Phys. Rev. Lett. **81**, 5644 (1998).

[19] P.J. Kortman and R.B. Griffiths, Phys. Rev. Lett. **27**, 1439 (1971).

[20] X.Z. Wang and J.S. Kim, Phys. Rev. E **59**, 1242 (1999).